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15 december — Midtoets Complexe Analyse

1a $f(z)=x^2+iy^2=u(x,y)+iv(x,y)$ with $u(x,y)=x^2$ and $v(x,y)=y^2 \Rightarrow \frac{\partial u}{\partial x}=2x,\ \frac{\partial v}{\partial y}=2y,\ \frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial x}$

Then the Cauchy-Riemann equations are not satisfied: $\frac{\partial u}{\partial x}=2x\neq 2y=\frac{\partial v}{\partial y}$ for at least some x and y, so the function is not differentiable for all x and y, thus also not analytic on $\mathbb{C}$.

b On the line $x=y$, $\frac{\partial u}{\partial x}=2x=2y=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=0=-\frac{\partial v}{\partial x}$, so the Cauchy-Riemann equations are satisfied, and all partial derivatives are continuous, so f is differentiable on $x=y$.

2a $f(z)=\cos z=\frac{e^{iz}+e^{-iz}}{2}=0 \Rightarrow e^{iz}=-e^{-iz}=e^{i\pi}e^{-iz}=e^{i(\pi-z)}$

$\Rightarrow iz=i(\pi-z)+2k\pi i=i(-z+(2k+1)\pi) \Rightarrow z=-z+(2k+1)\pi$

$\Rightarrow 2z=(2k+1)\pi$

$\Rightarrow z=\frac{2k+1}{2}\cdot\pi$ $\Big\}$ with $k=0,1,2\ldots$

k and π are real numbers, so z is always real if it's a zero of $\cos z$, too.

b $f(z)=\cos z=\frac{e^{iz}+e^{-iz}}{2}=\frac{e^{x+iy}+e^{-x-iy}}{2}=\frac{e^{x}e^{iy}+e^{-x}e^{-iy}}{2}=\frac{e^{x}(\cos y+i\sin y)+e^{-x}(\cos -y+i\sin -y)}{2}=\frac{e^{x}(\cos y+i\sin y)+e^{-x}(\cos y-i\sin y)}{2}$

$=\frac{e^{x}\cos y+e^{-x}\cos y}{2}+i\,\frac{e^{x}\sin y-e^{-x}\sin y}{2}=\underbrace{\frac{\cos y\,(e^{x}+e^{-x})}{2}}_{u(x,y)}+i\,\underbrace{\frac{\sin y\,(e^{x}-e^{-x})}{2}}_{v(x,y)}=u(x,y)+iv(x,y)$

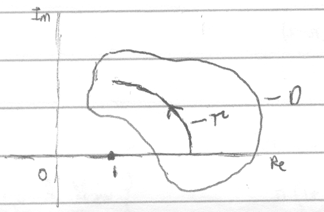
Cauchy-Riemann eq. 1: $\frac{\partial u}{\partial x}=\frac{\cos y}{2}(e^{x}-e^{-x})=\cos y\,\frac{(e^{x}-e^{-x})}{2}=\frac{\partial v}{\partial y}$

Cauchy Riemann eq. 2: $\frac{\partial u}{\partial y}=-\sin y\cdot\frac{(e^{x}+e^{-x})}{2}=-\left(\frac{\sin y\,(e^{x}+e^{-x})}{2}\right)=-\frac{\partial v}{\partial x}$

All partial derivatives are continuous, so f is differentiable on the whole complex plane $\Rightarrow$ f is analytic on the open set $\mathbb{C}$ $\Rightarrow$ f is entire.

2c No. Each function that's bounded and analytic on an open set is constant. But our function is not constant (see 2a, it has zeros but also non-zeros) so it cannot be analytic and bounded. But it is analytic (for it is entire, see 2b) so it cannot be bounded.

3a $\frac{1}{z-1}$ is analytic on $\mathbb{C} \setminus \{1\}$. So the following drawing can be made:



with D ~~as~~ a simply ~~conn~~ connected domain where $\frac{1}{z-1}$ is analytic. So we can ~~use~~ use (with $f = \frac{1}{z-1}$)

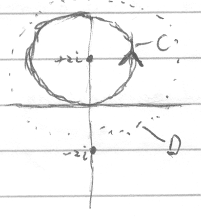
$\int_\Gamma \frac{dz}{z-1} = \int_\Gamma f(z)\,dz = F(1+i2) - F(1+2)$ $\quad f = (z-1)^{-1} \Rightarrow F = \cancel{\ } \log(z-1)$

$\Rightarrow \int_\Gamma \frac{dz}{z-1} = \log(i\cdot 2) - \log(2) = \mathrm{Log}|2| + i\,\mathrm{Arg}(i\cdot 2) + 2k_1\pi i - \mathrm{Log}|2| - i\,\mathrm{Arg}(2) - 2k_2\pi i = i(\tfrac{1}{2}\pi + 2\pi(k_1 - k_2))$

We have to take the principal value of each log to get 1 definite answer, so $k_1 = k_2 = 0$ gives

$\int_\Gamma \frac{dz}{z-1} = \frac{1}{2}\pi i$

4 $\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$ so $\oint \frac{e^{-z}}{z^2+4}$ has poles at $-2i$ and $+2i$.



Only $+2i$ lies in the contour over which we integrate, so $\frac{e^{-z}}{z^2+4}$ is analytic on the connected domain $D \setminus \{2i\}$.

Cauchy's integral theorem says $f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz$. If we take $f(z) = \frac{e^{-z}}{z+2i}$ and $z_0 = 2i$, then this becomes:

$$2\pi i f(z_0) = \int_\Gamma \frac{f(z)}{z-z_0}\,dz \Rightarrow 2\pi i \frac{e^{-2i}}{4i} = \int_\Gamma \frac{e^{-z}}{(z+2i)(z-2i)}\,dz = \int_\Gamma \frac{e^{-z}}{z^2+4}\,dz$$

$\Gamma = C$, thus $\int_C \frac{e^{-z}}{z^2+4}\,dz = 2\pi i \frac{e^{-2i}}{4i} = \frac{\pi}{2} e^{-2i}$.

compact

5 a ~~must be~~ The maximum value theorem says that each function on a closed set should take a maximum somewhere or multiple times (or be constant), and $|f(z)|$ is also a function so there should be at least one maximum on V. This maximum will be somewhere on the boundary of V, because $f(z)$ is analytic on V and V is a simple connected domain.

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b M must be attained on the boundary of V (for f is analytic on some domain containing V), thus on one of the line segments $0 \to 1$, $1 \to 1+i$, $1+i \to i$ and $i \to 0$. M can be found by finding the maxima m_1, m_2, m_3 and m_4 on the 4 segments and comparing these.

m_1: on the segment $0 \to 1$ $|f|$ is monotonously decreasing (for $|(z-2)|$ and $|(z-3)|$ decrease monotonously, so there product does, too). So $m_1 = |f(0)| = 6$

m_2: on the segment $1 \to 1+i$, $|f|$ is monotonously increasing. This is true because: ~~is monotonously~~

$|(z-2)| = |\underbrace{x-2}_{\text{constant}} + \underbrace{iy}_{\text{increasing}}|$, $|(z-3)| = |\underbrace{x-3}_{\text{constant}} + \underbrace{iy}_{\text{increasing}}|$, so $|(z-2)|$ and $|(z-3)|$ increase monotonous, so there product does, too.

So $m_2 = |f(1)| = 2$

m_3 : from $1+i$ to i, $|f(x)|$ is monotously increasing. This is true because

$(|x-2|) = |x-2+iy| = |x-2+\underbrace{iy}_{\text{constant}}|$, $(|x-3|) = |x-3+iy| = |x-3+\underbrace{iy}_{\text{constant}}|$, the i's are constant and $x-2$ and $x-3$ are monotonous decreasing (and smaller than 0). So $|x-2|$ and $|x-3|$ are monotonous increasing, thus there product too.

$m_3 = |f(i)| = |(i-2)(i-3)| = |-1+6-5i| = \sqrt{5^2+(-5)^2} = \sqrt{50} = 5\sqrt{2}$

m_4 : from i to 0, $|f(x)|$ is monotonously decreasing (because $|x-2|$ and $|x-3|$ are monotously decreasing, there product is too).

So $\cancel{m_4 = |f(i)|}$ $m_4 = |f(i)| = 5\sqrt{2}$

Thus $M = \max(m_1, m_2, m_3, m_4) = 5\sqrt{2}$